GENERALIZATION OF CHROMATIC NUMBER

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Introduction

Unless otherwise specified, graphs considered here are finite, undirected, loopless and without multiple edges. The *chromatic number* of a graph G, denoted by $\chi(G)$, is defined as the least number of colors needed to color the vertices of G such that no two adjacent vertices get the same color. Equivalently, it is the least integer k for which there exists a partition of the vertex set of G into k subsets $V_1, V_2, \ldots, V_k$ such that each induced subgraph $<V_i>$ is an independent set. One generalization of this concept is given in the following definition.

Definition 1.1 Let H be a graph. The *H-chromatic number of G*, denoted by $\chi(G; H)$, is defined to be the least interger k for which there exists a partition of the vertex set of G into k subsets $V_1, V_2, \ldots, V_k$ such that each connected component of $< V_i >$ is an induced subgraph of H.

We observe that if H is the trivial graph with one vertex, then the H-chromatic number of G coincides with the usual chromatic number of G.

Theorem 1.1. If H_1 is an induced subgraph of H_2, then for any graph G, $\chi(G, H_1) \geqslant \chi(G; H_2)$.

Proof: Let $k = \chi(G; H_1)$. Then there exists a partition of the vertex set of G into k subsets $V_1; V_2, \ldots, V_k$ such that the connected components of $< V_i >$ are induced subgraphs of H_1. It follows that these connected components are also induced subgraphs of H_2. Therefore $\chi(G; H_2) \leqslant k$.

In the next section, we shall focus on H-chromatic number of graphs, where H is a path (finite or infinite in order).

Path Chromatic Number

Let H be a path of order k. We shall call $\chi(G; H)$ the *k-path chromatic number* of G, and we shall denote it by $\chi(G; P_k)$. Observe that $\chi(G; P_1) = \chi(G)$, the usual chromatic number of G. It follows from Theorem 1.1 that

$$(*) \quad \chi(G; P_1) \geqslant \chi(G; P_2) \geqslant \chi(G; P_3) \geqslant \chi(G; P_4) \geqslant \cdots$$

The proof of the following theorem can be found in [1].

Theorem 2.1. Let G be a graph with maximum degree d. Then for each $k \geq 2$,

$$\chi(G; P_k) \leq \left\lceil \frac{d+1}{2} \right\rceil$$

It is well known that for any planar graph G, $\chi(G) \leq 4$ and that this bound is best possible. It is quite natural to ask for the best upper bound for $\chi(G; P_k)$, where $k \geq 2$. Intuitively, one would expect an upper bound less than 4.

Let us consider first the case of outer planar graphs. It is known that $\chi(G) \leq 3$ for all outer planar graphs and that this bound is the best possible. How about $\chi(G; P_k)$, where $k \geq 2$?

Theorem 2.2. If G is outer planar and $k \geq 2$, then $(G; P_k) \leq 3$ and this bound is best possible.

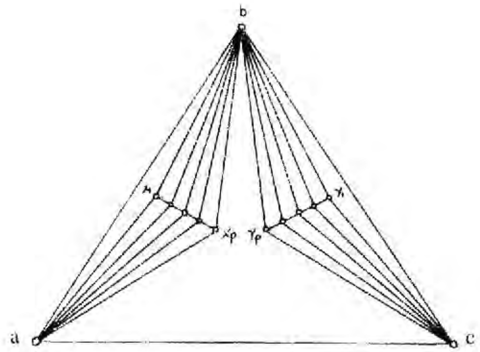
Proof: We shall use mathematical induction on the order of G. If the order of G is $n = 1$, 2 or 3, the theorem is easily seen to be true. Let $n \geq 4$ and assume that the theorem holds for all outer planar graphs or order $n - 1$. Choose a vertex v of degree at most 2 and let $G' = G - v$. By induction hypothesis, there exists a coloring of the vertices of G' using at most 3 colors such that vertices having the same color induce a graph all of whose connected components are paths of order not exceeding k. Since v has at most two neighbors, we can color it using a color out of the 3 colors such that it is differently colored from any of its neighbors. Hence, $\chi(G; P_k) \leq 3$. This is best possible upper bound since the outer planar graph $G = K_1 + P_{2k+2}$ has k-path chromatic number equal to 3.

Now let us consider planar graphs in general. We know that there exist planar graphs G for which $\chi(G) = 4$. What about $\chi(G; P_k)$, where $k \geq 2$?

Theorem 2.3. For each $k \geq 2$, there exists a planar graph G for which $\chi(G; P_k) = 4$.

Proof: Let $p = 5k + 5$ and let $I = x_1 x_2 \ldots x_p$ and $J = y_1 y2 \ldots y_p$ be two vertex-disjoint paths. Let $K_3 = \{a, b, c\}$ be a clique without vertices in common with I or J. Form the planar graph $G = (\{a, b\} + I) \cup (\{b, c\} + J)$ shown at the following page.

By the Four-Color Theorem, we know that $\chi(G; P_k) \leq 4$. Now, suppose that $\chi(G; P_k) \leq 3$. Since $\{a, b, c\}$ is a clique, the vertices a, b, c cannot all have the same color. Without loss of generality, let us assume that the vertex a has

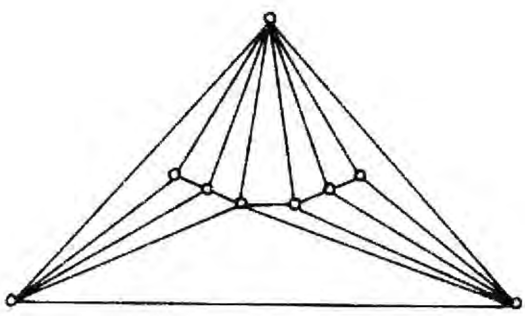


color 1 and that vertex b has color 2. Then at most two vertices in I have color 1 and also at most two vertices (in I) have color 2. Since I has $5k + 5$ vertices, then at least $5k + 1$ vertices in I have color 3. This implies the existence of at least one path in I of order greater than k all of whose vertices are colored 3. This is a contradiction. Hence, $\chi(G; P_k) = 4$.

Let H be a path of infinite order. Then we shall denote $\chi(G; H)$ by the symbol $\chi(G; P_\infty)$. In view of (*), one would expect $\chi(G; P_\infty)$ to be strictly less than $\chi(G)$. For outer planar graphs, this is indeed true. A proof of the next theorem can be found in [1].

Theorem 2.4. If G is an outer planar graph, then $\chi(G; P_\infty) \leqslant 2$.

For a planar graph in general, 2 is not the correct upper bound for $\chi(G; P_\infty)$. The following planar graph has $\chi(G; P_\infty) = 3$.



It is not known, however, if 3 is the best upper bound for $\chi(G; P_\infty)$ for planar graphs G.

Other Generalizations

Evidently, the concept of chromatic number can be generalized in many different ways and path chromatic number is one of these. By simply specifying the graph H in Definition 1.1, we get a new and generalized concept of chromatic number which will always coincide with the usual concept of chromatic number when H is the trivial graph with only one vertex. However, it is convenient to choose H to be a special graph – one with a not so complicated structure. For example, we take H to be the star S_k $(k \geqslant 0)$ and call the associated number the star chromatic number. Other examples of graphs H we can use are the complete graph, the empty graph (the complement of the complete graph), etc.

References

[1] Akiyama, J., H. Era and S. Gravacio. 1986. "Path Chromatic Number and Path Stable Sets", *Proceedings of the First China-U.S.A. Conference in Graph Theory and its Applications*, China.

[2] Harary, F. 1969. *Graph Theory*. Addison-Wesley, Reading.