A SIMPLE SOLUTION TO THE STROTZ CONSISTENCY PROBLEM

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ABSTRACT

As an alternative to the Pollak solution to the Strotz problem of finding a time consistent optimal plan, there is a simple Strotz-type solution that uses appropriate relative weights between the present and the future.

Introduction

In a celebrated paper Strotz (1956) raised the problem of formulating an optimal plan that is time consistent, i.e. one whose continuation would be optimal at any later time during the plan period. Strotz gave a solution which was, however, considered faulty by Pollak (1968) who proposed an ingenious though somewhat cumbersome procedure. Apparently Strotz accepted Pollak's criticism, but there is a simpler solution which follows the lines of Strotz' original discussion.

The Strotz Model

Let the "instantaneous utility function" $u(x)$ be given with $u'(x) > 0$ and $u''(x) < 0$. Denoting consumption at time t by $x(t)$, assume that a person's preferences at time τ would make him maximize the utility functional

$$(1) \quad \phi_\tau = \int_\tau^T w(t-\tau)\; u(x(t))\, dt \qquad 0 \leqq \tau \leqq T$$

subject to the conditions

$$(2) \quad \int_\tau^T x(t)\, dt = K(\tau)$$

$$(3) \quad K(\tau) = K(0) - \int_0^\tau x(t)\, dt$$

where the initial stock $K(0)$ is given and $\int_0^\tau x(t)dt$ is a historical fact at the decision point τ. It is the discount or weight function $w(t-\tau)$, which shifts the discounting of the future with τ, that creates the Strotz problem. Writing $v(x) = u'(x)$ we know that for a maximum of (1) it is necessary that

$$(4) \quad \dot{v}(x(t))/v(x(t)) = -\dot{w}(t-\tau)/w(t-\tau), \; 0 \leqq \tau \leqq t \leqq T.$$

Consider the planned consumption path over time that is formulated at $\tau = 0$. According to (4), the plan must satisfy

$$(5) \quad \dot{v}(x(t'))/v(x(t')) = -\dot{w}(t')/w(t')$$

at time $t' > 0$. Suppose a re-evaluation of the plan is made when that time comes. With $\tau = t'$, (4) now requires that

$$(6) \quad \dot{v}(x(t'))/v(x(t')) = -\dot{w}(0)/w(0).$$

In general the right-hand sides of (5) and (6) would be unequal, in which case time inconsistency is said to arise: the optimal plan formulated at t' is not a continuation of the original plan.

A "rational" person able to foresee such an inconsistency would precommit his future decisions or, failing that possibility, adopt a consistent plan. Under a strategy of precommitment a person would "try to ensure that he will do tomorrow that which is best from the standpoint of today's desires," and if that is not feasible, under a strategy of consistent planning one will "reject any plan which he will not follow through. His problem is then to find the best plan among those that he will actually follow" (Strotz 1956, p. 173).

Where precommitment is not possible, Strotz showed that without additional assumptions a consistent plan (which will need no revision at any later decision point) requires the discount function to be exponential specializing (1) to

$$(1') \quad \psi_\tau = \int_\tau^T e^{-r(t-\tau)} u(x(t))\,dt, \; r = \text{const.}$$

In the Strotz framework, this is permitted: the original discount function $w(t - \tau)$, which typically overvalues "the more proximate satisfactions relative to the more distant ones" (p. 177), can be replaced in such a way that all future dates get discounted at a constant rate. Strotz then argued that

$$(7) \quad r = -\dot{w}(0)/w(0)$$

in (1′) because of (6), which Pollak has questioned.

The Pollak Solution

Pollak's (1968) approach retains the original maximand (1) but imposes constraints at every decision point so that the constrained optimal path is followed through. Specifically, suppose that plan re-evaluations are made at the predesignated points $\tau_1, \ldots, \tau_n$ with $\tau_1 = 0$ and $\tau_n < T$. Instead of $x(t)$ write $x(t; K(\tau_i))$ which is restricted by the condition

$$\int_{\tau_i}^{T} x(t, K(\tau_i))\, dt = K(\tau_i)$$

Let $\left\{ x^*(t; K(\tau_n)) \right\}_{\tau_n}^{T}$ denote the consumption path which at τ_n is most preferred among all paths $\left\{ x(t; K(\tau_n)) \right\}_{\tau_n}^{T}$ from τ_n to T. That preferred path is of course a function of $K(\tau_n)$. Next, define $\left\{ x^*(t; K(\tau_{n-1})) \right\}_{\tau_{n-1}}^{T}$ as most preferred at τ_{n-1} among all paths $\left\{ x(t; K(\tau_{n-1})) \right\}_{\tau_{n-1}}^{T}$ which have subpaths

$$(8)\quad \left\{ x(t; K(\tau_{n-1})) \right\}_{\tau_n}^{T} = \left\{ x^*(t; K(\tau_n)) \right\}_{\tau_n}^{T}$$

where

$$K(\tau_n) = K(\tau_{n-1}) - \int_{\tau_{n-1}}^{\tau_n} x(t; K(\tau_{n-1}))\, dt.$$

Putting $i = n$, further recursion leads finally to a definition of the path $\left\{ x^*(t; K(0)) \right\}_{0}^{T}$ which, by construction, is followed through at each decision point τ_i $(i = 2, \ldots, n)$. Letting $n \to \infty$, Pollak assumes that a limit path exists[1] which he proposes as the solution to the Strotz problem of finding a consistent plan.

In the special case $u(x) = \log x$, Pollak shows that the limit path would coincide with the "naive" path, i.e. the path that would be traced by maximizing (1) at every τ. Since the naive path turns out to be different from the Strotz solution implied by (7) in (1′), Pollak concluded that the latter is incorrect.

Observe that the Pollak argument is valid only if the limit path is best among all consistent plans, but it is not clear why this should be so if the maximand can be changed to (1′). Notice also that the Pollak solution is time consistent only because at every decision point it is assumed that future decisions will be constrained to subsets of possible plan continuations e.g. those satisfying (8). This means that some kind of precommitment is possible which obviates the need for a consistent plan in the Strotz schema. One could argue therefore that while Pollak's "backward optimization" procedure is interesting in itself,[2] its feasibility implies its own redundancy.

An Alternative Solution

The Strotz problem of finding a consistent optimal plan arises only when precommitment is not feasible, in which case the only way to have a consistent plan is to replace (1) by (1′) at the start. The remaining question then is the "correct"

[1] It is not all clear that a limit path exists. Even in the n decision point case, there is an existence and uniqueness problem about the Pollak solution; see Peleg and Yaari (1973) and Goldman (1980).

[2] See e.g. Hammond (1976) in addition to the references in footnote 1.

value of r, and Strotz does go wrong with (7). Remembering that (6) is based on (1) as the maximand, there is no necessity for (7) after (1′) has replaced (1). In fact, (7) runs counter to Strotz' own strictures against the myopia of overvaluing the near future (emphasized by the title of his paper), and it fails to use the one property of the discount function w that seems most relevant in the nature of the case.

A persons's original w gives the specific distribution of weights that he assigns over the plan period, but having changed that distribution to an exponential one in order to have a consistent plan, it is only reasonable to give the right weight to the future in making his initial decision. Accordingly, let a satisfy

$$(9) \quad w(0)/\int_0^T w(t)\,dt = 1/\int_0^T e^{-at}\,dt.$$

Then, putting $r = a$ in (1′) gives him a plan where (4) simply becomes

$$(4') \quad \dot{v}(x(t))/v(x(t)) = a \qquad\qquad 0 \leqq t \leqq T$$

so the new plan at any later time is merely the continuation of the original one.

Concluding Remark

A straightforward solution to the Strotz problem can be obtained by meeting its original requirements and assigning appropriate weights to the present and the future in accordance with the original discount function. A fundamental question that could be raised, however, is whether the concept of a discount function is necessary or even useful for an analysis of decisions over time. For an alternative approach that dispenses with a discount function even with infinite time horizons, see Encarnación (1983).

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