Trans. Nat. Acad. Sci. & Tech. (Phils.)
1987.9:43-50

ON THE LEXICOGRAPHIC OF THE n-PERMUTATIONS

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ABSTRACT

We call $\pi = x_1 x_2 \ldots x_n$ an *n–permutation* of the elements $1, 2, \ldots, n$ if each x_1 is a positive integer not exceeding n and $x_i \neq x_j$ for all $i \neq j$. We refer to x_1 as the *ith element* of the permutation π. If $\pi_1 = x_1 x_2 \ldots x_n$ and $\pi_2 = y_1 y_2 \ldots y_n$ are two n–permutations, we say that π_1 *precedes* π_2 if there exists a positive interger k not exceeding n such that $x_i = y_i$ for all $i < k$ and $x_k < y_k$. If no such integer k exists, then $x_i = y_i$ for all i and we say that the permutations are *equal*. We shall write $\pi_1 < \pi_2$ if π_1 precedes π_2, and $\pi_1 = \pi_2$ if they are equal. We know that there are exactly n! permutations on $1, 2, \ldots, n$. It is easy to see that for any two n-permutations π_1 and π_2, exactly one of the following holds: $\pi_1 = \pi_2$, $\pi_1 < \pi_2$, $\pi_2 < \pi_1$. Furthermore, if $\pi_1 < \pi_2$ and $\pi_2 < \pi_3$ then $\pi_1 < \pi_3$. Thus, $<$ is a linear ordering on the n-permutations. This is called the *lexicographic ordering*.

If exactly r distinct permutations precede π, we say that π has *rank* r + 1. Thus, 123 . . . n has rank n and n . . . 321 has rank n! We shall denote by rank (π) the rank of π.

Three main problems are dealt with in this paper. The first is the computation of the rank of a given permutation, the second is the determination of the permutation with a given rank, and the third is the lexicographic enumeration of n-permutations.

The Rank of an n-Permutation.

For convenience, let us define the *index* of the permutation $\pi = x_1 x_2 \ldots x_n$ to be $\mathrm{ind}(\pi) = \tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_n$, where $\tilde{x}_i$ is the number of subscripts $j > i$ such that $x_j < x_i$. Clearly, $0 \leqslant \tilde{x}_1 \leqslant n-i$ for each i. In particular, $\tilde{x}_n = 0$. As an example, if $\pi = 316425$, then $\mathrm{ind}(\pi) = 203100$. It should be remarked that if $n > 9$, then we should use the notation $\pi = (x_1, x_2, \ldots, x_n)$ and $\mathrm{ind}(\pi) = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ to avoid any confusion.

Theorem 2.1. *If $\pi = x_1 x_2 \ldots x_2$ is an n-permutation, then*

$$rank(\pi) = 1 + \sum_{k-1}^{n-1} \tilde{x}_k (n-k)!$$

Proof: Let $\pi' = y_1 y_2 \ldots y_n$ be any permutation which precedes π. Let us count the total number of such permutations π' as follows. Let k be a positive integer less than n. Set $y_1 = x_1$ for all $i < k$. Choose any $j > i$ such that $x_j < x_k$ and set $y_k = x_j$. Note that we have a total of $\widetilde{x}_k$ choices here. For the last n–k elements of π', we choose any of the (n–k)–permutations of $[1, 2, \ldots, n] / [y_1, y_2, \ldots, y_k]$. Hence, the number of such permutations π' we can construct corresponding to k is $\widetilde{x}_k(n-k)!$. Clearly, two distinct values of k will give us non-overlapping sets of permutations that precede π. The theorem then follows.

Theorem 1 solves our first problem. As an illustration, if k = 316425, then ind(π) = 203100 and rank(π) = 1 + 2(5!) + 0(4!) + 3(3!) + 1(2!) + 0(1!) = 261. Thus, the permutation 316425 is the 261*st* permutation in the lexicographic list of all 6-permutations.

Finding the permutation with a given rank.

Lemma 3.1. *Distinct n-permutations have distinct indices.*

Proof: Let $\pi_1 = x_1 x_2 \ldots x_n$ and $\pi_2 = y_1 y_2 \ldots y_n$ be distinct n-permutations. Without loss of generality, we may assume $\pi_1 < \pi_2$. Then there exists a positive integer $k < n$ such that $x_k < y_k$ and $x_1 = y_1$ for all $i < k$. Let $j > k$ and $x_j < x_k$. Then $x_j = y_t$ for some $t > k$ and $y_t < y_k$. It follows that $\widetilde{y}_k \geqslant \widetilde{x}_k$. But in addition, $x_k = y_s$ for some $s > k$ and $y_s < y_k$. Therefore $\widetilde{y}_k > \widetilde{x}_k$ and hence ind(π_1) $\neq$ ind(π_2) since they differ in their k*th element.*

Corollary. *Let $a_1 a_2 \ldots a_n$ be an ordered n-tuple of integers such that $0 \leqslant a_1 \leqslant n-i$ for each i. Then there exists a unique n–permutation π with ind(π) = $a_1 a_2 \ldots a_n$.*

Proof: Consider the set P_n of all ordered n-tuples of integers $c_1 c_2 \ldots c_n$ where $0 \leqslant a_1 \leqslant n-i$ for each i. Note that P_n has exactly n! elements. By the theorem, the mapping ind from the set of all permutations of 1, 2, . . . , n to P_n is one-to-one. It follows that ind is also an onto map since there are exactly n! n–permutations. Hence, any element $a_1 a_2 \ldots a_n$ of P_n is the index of a unique permutation π.

Although the above argument proves the existence of a unique permutation with a given index, it does not show us a way of constructing the permutation. We shall now develop an algorithm for constructing the permutation with a given index.

Let $a_1 a_2 \ldots a_n$ be a given index. Let $C_i = c_1^1 c_2^1 \ldots c_n^1 = 123 \ldots n$. Since $0 \leqslant a_i \leqslant n-i$, then $i \leqslant a_i + i \leqslant n$ and hence $1 \leqslant a_i + 1 \leqslant n-i+1$ for each i. Define x_1 to be the $(a_i + 1)$ *th* element of C_i and let $C_2 = c_2^2 c_2^2 \ldots c_{n-1}^2$ be the (n–1)–tuple obtained from C_1 by striking out the element x_1, the $(a_1 + 1)$ *th* element. Recursively, we define $C_i = c_1^1 c_2^1 \ldots c_{n-1+2}^1$ to be the ordered (n–i+1)–tuple obtained from C_{i-1} by striking x_{i-1} out. Since the elements of C_i form an increasing sequence, it is clear that the elements of C_i also form an increasing sequence, for each i. Let x_1 be the (a_i+1) *th* element of C_i and form the permutation $\pi = x_1 x_2 \ldots x_n$. We claim that $\text{ind}(\pi) = a_1 a_2 \ldots a_n$. Now, since x_i is the (a_i+1) *th* element of C_i and the element of C_i form a non-decreasing sequence, it follows from the construction of π that the number of elements x_j of π such that $j > i$ and $x_j > x_i$ is equal to the number of elements of C_1 to the left of the (a_1+1)*th* element. Hence, $\widetilde{x_i} = a_i$ and $\text{ind}(\pi) = a_1 a_2 \ldots a_n$.

Lemma 3.2. *Every positive integer r not exceeding n! can be expressed uniquely in the form* $r = 1 + \sum_{i=1}^{n-1} a_i(n-i)!$, *where* $0 \leqslant a_i \leqslant n-i$ *for each i.*

Proof: Let $1 \leqslant r \leqslant n!$. Then r is the rank of a unique n–permutation $\pi = x_1 x_2 \ldots x_n$. Let $\text{ind}(\pi) = a_1 a_2 \ldots a_n$. Then $0 \leqslant a_i \leqslant n-i$ for each i and by Theorem 2.1, $r = 1 + \sum_{i=1}^{n-1} a_i(n-i)!$. To prove the uniquess of this representation, let us suppose $r = 1 + \sum_{i=1}^{n-1} b_i(n-i)!$, where $0 \leqslant b_i \leqslant n-i$ for each i. Let $b_n = 0$. By the corollary to Lemma 3.1, there exists a unique n–permutation π' with $\text{ind}(\pi') = b_1 b_2 \ldots b_n$. Bur rank $(\pi') = r = \text{rank}(\pi)$. It follows that $\pi = \pi'$ since there is a unique permutation with a given rank. From Lemma 3.1, it follows that $b_i = a_i$ for each i.

Theorem 3.1. *Let r be an integer such that* $1 \leqslant r \leqslant n!$. *Let* $r_o = r-1$ *and for each* $i < n$, *let* q_i *and* r_i *be the quotient and remainder, respectively, obtained by dividing* $(n-i)!$ *by* r_{i-1}. *Then* $r = 1 + \sum_{i=1}^{n-1} q_i(n-i)!$ *and* $0 \leqslant q_i \leqslant n-i$ *for each i.*

Proof: Consider the equation $r_{1-2} = q_i\,(n-i)! + r_i$. Suppose that $q_i > n-i$. Then $q_i \leqslant n-i+1$). This implies that $r_{i-2} \leqslant (n-1)!\,(n-i+1) + r_1 = (n-i+1)! + r_i \geqslant (n-i+1)!$ This is a contradiction since $r_{i-1} < (n-1+1)!$.

Now, if we add all the equations $r_{i-1} = q_i\,(n-i)! + r_i$, where $i = 1, 2, \ldots, n-1$, then we get

$$r_o + \sum_{i=1}^{n-2} r_i = \sum_{i=1}^{n-1} q_i\,(n-i)! + \sum_{i=1}^{n-1} r_i,$$

which implies that

$$r_o = \sum_{i=1}^{n-1} q_i\,(n-1)! + r_{n-1}$$

But $r_{n-1} = 0$ since it is the remainder obtained by dividing r_{n-2} by $1!$. The theorem then follows since $r_o = r - 1$.

Corollary. *Let* $1 \leqslant r \leqslant n!$ *and let* $q_1, q_2, \ldots, q_{n-1}$ *be the integers defined in the theorem. Then* $q_1 q_2 \ldots q_n$, *where* $q_n = 0$, *is the index of the permutation* π *whose rank is r.*

Example. Find the 6–permutation with rank r = 15. First we determine the quotients q_i.

$$14 = 0(5!) + 14 ===> q_1 = 0,$$
$$14 = 0(4!) + 14 ===> q_2 = 0,$$
$$14 = 2(3!) + 0 ===> q_3 = 2,$$
$$2 = 1(2!) + 0 ===> q_4 = 1,$$
$$0 = 0(1!) + 00 ===> q_5 = 0,$$
$$\text{and set} \quad q_6 = 0.$$

Therefore ind(π) = 002100. Let C_1 123456. Using our algorithm, we have

$$0 + 1 = 1 ===> x_1 = 1st \text{ element of } C_1 = 1 ===> C_2 = 23456$$
$$0 + 1 = 1 ===> x_2 = 1st \text{ element of } C_2 = 2 ===> C_3 = 3457$$
$$0 + 1 = 3 ===> x_3 = 3rd \text{ element of } C_3 = 5 ===> C_5 = 346$$
$$1 + 1 = 2 ===> x_4 = 2nd \text{ element of } C_4 = 4 ===> C_3 = 36$$
$$0 + 1 = 1 ===> x_5 = 1st \text{ element of } C_5 = 3 ===> C_6 = 6$$
$$0 + 1 = 1 ===> x_6 = 1st \text{ element of } C_6 = 6$$

Hence π = 125436 is the 6–permutation with rank 15.

It should be observed that the computation of the quotients q_i is not easy if n is large. A much easier method of computation is possible. Suppose $1 \leqslant r \leqslant n!$. Let $r_o = r - 1$ and let the results of the series of divisions be the following.

$$r_o = q_1 \ (n-1)! + r_1 \qquad 0 \leqslant r_1 \ \mathrm{H}$$
$$r_o = q_2 \ (n-1)! + r_1 \qquad 0 \leqslant r_1 < (n-1)!$$
$$r_1 = q_2 \ (n-2)! + r_2 \qquad 0 \leqslant r_2 < (n-2)!$$
$$r_2 = q_3 \ (n-3)! + r_3 \qquad 0 \leqslant r_3 < (n-3)!$$

$$r_{n-2} = q_{n-1} \ (1!) + r_{n-1} \qquad 0 \leqslant r_{n-1} < 1!$$

We shall use the fact that for any real number x and any positive integer n, $[x/n] = [\,[x]/n]$, where $[x]$ denotes the greatest integer not exceeding x (This is not so difficult to prove.).

Consider the following sequence of divisions.

$$r_o = q_1'(1) + r_1 \qquad 0 \leqslant r_1' < 1$$
$$q_1' = q_2'(2) + r_2' \qquad 0 \leqslant r_2' < 2$$

$$q_2' = q_3'(3) + r_3' \qquad 0 \leqslant r_3' < 3$$

$$\vdots$$

$$q_{n-1}' = q_n'(n) + r_n' \qquad 0 < r_3' < n$$

Observe that $q_1' = [r_o/1]$, $q_2' = [q_1'/2] = [r_o/2!]$, $q_3' = [q_2'/3] = [r_o/3!]$, . . . , $q_{n-1}' = [r_o/(n-1)!]$, $q_n' = [r_o/n!] = 0$. Multiply the second equation by 1!, the third by 2!, and in general multiply equation i + 1 by i!. If all the resulting equations are added, we get $q_1' = q_n'(n1] + \sum_{i=1}^{n-1} r_{n-1+1}' (n-i)!$. But $q_1' = r_o$ and $q_n' = 0$. Furthermore, $0 \leqslant r_{n-1+2}' < n-i+1$. By Lemma 3.2 (uniqueness), it follows that $r_{n-i+1}' = q_i$ for each i = 1, 2, . . . , n–1.

Example. Represent 15 in the form $1 + \sum_{1=1}^{n-1} q_i (n-1)!$, where $0 \leqslant q_1 \leqslant n-i$ for all i.

We have $r_o = 14$. Then we perform the following divisions.

14/2 = 7,	remainder 0,	$q_5 = 0$
7/3 = 2,	remainder 1,	$q_4 = 1$
2/4 = 0,	remainder 2,	$q_3 = 2$
0/5 = 0,	remainder 0,	$q_2 = 0$
0/6 = 0,	remainder 0,	$q_1 = 0$

So we have 15 = 1 + 0(5!) + 0(4!) + 2(3!) + 1(2!) + 0(1!).

4. Lexicographic Enumeration of n–Permutations.

There are five methods of enumerating n–permutations discused in [1], namely, the Tompkins-Paige method, the derangement method of M. Hall, the transposition method of M.B. Wells, the adjacent mark method of S. M. Johnson, and the lexicographic method of D. N. Lehmer. Of these methods, only Lehmer's method lists the n–permutations in lexicographic order. Essentially the method consists of listing the n–permutations starting from the one with rank 1, i.e., the permutation 123. . . n, and then incrementing the rank by one until the last permutation n . . . 321 is reached. For each given rank, the permutation is determined by means of a neat algorithm but the whole process takes time to complete because of the need to compute for the index of the permutation. Here we shall find a way to accomplishing the same task without the need for computing indices.

Let π_1 and π_2 be n–permutations. We say that π_2 is the *successor* of π_1 if $\pi_1 < \pi_2$ and there exists no n–permutation π such that $\pi_1 < \pi < \pi_2$. Let $\pi_2 = y_1 y_2 \ldots y_n$ be the successor of $\pi_1 = x_1 x_2 \ldots x_n$. Then by definition, there exists a positive integer $k < n$ such that $x_i = y_i$ for all $i < k$ and $x_k < y_k$. Let $\pi = z_1 z_2 \ldots z_n$ be any n–permutation such that $\pi_1 < \pi$. Then there exists a positive integer $k' < n$ such that $x_i = z_i$ for all $i < k'$ and $x_{k'} < z_{k'}$. Clearly, $k' \leqslant k$. Therefore, to find the successor of an n–permutation π_1, we look for the n–permutation π_2 such that corresponding elements of π_1 and π_2 in all positions before the k*th*, for the maximum possible value of k.

Lemma 4.1. *Let $\pi_2 = y_1 y_2 \ldots y_n$ be the successor of $\pi_1 = x_1 x_2 \ldots x_n$ and let k be the positive integer satisfying $x_i = y_i$ for all $i < k$ and $x_k < y_k$. Then $x_k < x_{k+1}$, $y_1 < y_{i+1}$ for all $i > k$ and y_k is equal to the minimum x_j satisfying $j > k$ and $x_j > x_k$.*

Proof: Suppose $x_k > x_{k+1}$. Since $x_i = y_1$ for all $i < k$ and $x_k < y_k$, it follows that $y_k = x_j$ for some $j > k+1$. Hence, $x_{k+1}\ x_{k+2} \ldots x_n$ is not monotonic decreasing. Consequently, there exists a maximum $t >$ such that $x_t < x_{t+1}$. Let $\pi' = z_1 z_2 \ldots z_n$ be the permutation defined by $z_t = x_{t+1}$, $z_{t+1} = x_t$, and $z_i = x_i$ for all i different from t and t+1. Clearly, $\pi_1 < \pi$. But this is a contradiction since $t > k$. That $y_1 \nless y_{i+1}$ for all $i > k$ is proven using a similar argument. By definition of successor, y_k must be equal to the minimum x_j such that $x_j > x_k$ and $j > k$.

Lemma 4.1 gives us a way of constructing the successor of any given permutation, and hence, an algorithm to enumerate all permutations in lexicographic order. We illustrate this by means of an example. Let $\pi_1 = x_1 x_2 \ldots x_n = 261354$. The largest k for which $x_k < x_{k+2}$ is k = 4 corresponding to the element 3. The minimum element to the right of 3 which is greater than 3 is 4. Therefore, the successor of π_1 is $k_2 = 261435$.

A computer program in PASCAL that lists in lexicographic order all n–permutations is given below.

```
Program Permutation;
label       done, show;
type        values = 0..50;
var         i, j, k, order: values;
            perm: array [0..50] of values;

Procedure Search; { determine first element of permutation not equal to correspond-
ing element in the successor}
begin       i: = order;
            perm [0] : = 0;
            repeat
                  i: = i – 1;
           until (perm[i] < perm [i + 1]); [ith element will be changed if it is not
           equal to 0]
end;
```

```
Procedure SortTail {arrange in increasing order the elements to the right of the
ith}
label       okey;
ebgin       if i = order–1 then go to okay;
            for j: i + 1 to order– 1 do
                  for k: = j + 1 to order do
                        if perm[k] < perm[j] then
                        begin     perm[0] : = perm[k] ;
                                  perm[k] : = perm[j] ;
                                  perm[j]  : = perm [0] ;
                        end;
okay:
end;
Procedure Adjust; [interchange the ith element and the least element to its right and
greater than the ith]
begin       j: = i;
            repeat
                  j: = j + 1;
            until (perm[j] > perm[1] );
            perm[0] : = perm[j] ;
            perm [j]  : = perm [i] ;
            perm [i]  : = perm [0] ;
end;

Procedure Display;{write on the screen the permutation }
begin       for i: = 1 to order do
            write (perm [1] ,'');
            writeln;
end;


end;
begin       write (Enter the order (1–50): ' );
            readln (Order);
            for i: = 0 to order do {define the initial permutation 123. . . n }
                  perm(i): = i;
show:       display;
            search;
            if i = 0 then go to done;{end program if there is no more successor}
            sorttail;
            adjust;
            go to show;
done:       write ('End of the lexicographic list!');
end.
```

Reference

1. Beckenbach, E.F., ed., Applied Combinatorial Mathematics Wiley, 1966.