A COMBINATORIAL APPROACH TO SPIN MODELS

JOSE MARIA P. BALMACEDA
Department of Mathematics
College of Science
University of the Philippines
Diliman, 1101 Quezon City

ABSTRACT

The construction of new and powerful invariants of knots and links by Vaughan Jones in 1984 and its recent generalizations have led to the discovery of important connections between the theory of knots and other branches of mathematics and sciences. The new knot theory has already been useful to molecular biologists studying the double helices of DNA. In physics, models in statistical mechanics may be defined on a knot or link diagram so that a suitable variation of the partition function of the system is often a knot invariant. In 1989, Jones constructed spin models and posed the challenge of investigating combinatorial structures for sources of spin models. In this paper we present an approach, first observed by Francois Jaeger in 1992, to the study of spin models using a combinatorial object called an association scheme. We outline the background and method of this approach and prove several characterization theorems for spin models arising from some families of association schemes.

Keywords and phrases: association schemes, spin models, knots, links, combinatorics, statistical mechanics.

INTRODUCTION

In a 1989 seminal paper, Vaughan F. R. Jones introduced the concept of a spin model as a method to construct invariants of links in 3-space (Jones, 1989). The starting point of this new era in the study of knots and links was Jones' discovery in 1984 of his famous polynomial invariant, which was obtained in an entirely different setting, and for which he was awarded the 1990 Fields Medal.

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Jones pursued the connection between knot theory and statistical mechanics via the use of spin models. The spin models he discovered can be defined as pairs (W_+, W_-) of square complex matrices satisfying certain invariance equations, which then guarantee that the partition function (after an adequate normalization) defines a link invariant. Spin models are used to understand, for instance, phase transitions in statistical mechanics.

Since then the subject has been shown to be related to an amazingly wide variety of subjects such as von Neumann algebras, representations of semi-simple Lie algebras, finite groups, and more generally, statistical mechanics, topological field theory, quantum groups, among others. For instance, see (Jones, 1989), (Jaeger, 1992), (de la Harpe-Jones, 1993) and (Jaeger, 1995).

A major focus of Jones' work was directed towards the combinatorial aspect of the study and he posed the challenge of investigating combinatorial structures for sources of spin models. In his paper, Jones gave two examples of symmetric spin models and raised the question of finding new ones.

In 1992, Francois Jaeger found new spin models for evaluation of the Kauffman lin polynomial invariant using special association schemes (Jaeger, 1992). Association schemes are important combinatorial structures and are the main objects of study in algebraic combinatorics. They provide a unifying approach to the study of various mathematical objects such as algebraic graphs, codes, designs, finite geometries, and includes the theory of finite groups.

Indeed the question of finding new spin models has turned out to be intimately connected with the theory of association schemes. Many subsequent works, e.g. (Bannai-Bannai, 1993), (Nomura, 1994), (Jaeger, 1995), and (Bannai-Bannai, 1995), confirmed the importance of the following situation: the matrices of a spin model belong to the Bose-Mesner algebra of some self-dual association scheme, and can be obtained by solving a certain modular invariance equation associated with the character table of the scheme.

The simplest case is that of the Potts model for the Jones polynomial, which can be thought of as arising from the 1-class association schemes, i.e. complete graphs. We describe the model later using the language of association schemes.

Important work is now directed at showing that the connections between spin models, including subsequent generalizations like non-symmetric models and four-weight models by (Bannai-Bannai, 1993), (Kawagoe-Munemasa-Watatani, 1994) and (Bannai-Bannai, 1995), and the theory of association schemes is indeed general and non-arbitrary. A goal of this research direction is to obtain a classification of spin models in terms of association schemes.

This paper provides the background and methods of this combinatorial approach to the study of spin models. We also give a direct proof of the characterization of (generalized) spin models arising from non-symmetric conference graphs. This continues the author's work in (Balmaceda, 1993), where the symmetric case is treated.

In Section 2, we give a brief overview of the study of polynomial invariants of links, which was the main motivation of Jones' 1989 paper. We then discuss the connection between spin models and association schemes in Section 3. In the 4th Section we prove the main result characterizing the spin models arising from non-symmetric conference digraphs. We conclude the paper with some remarks about the directions of this research.

INVARIANTS OF KNOTS AND LINKS

By a link we mean a finite disjoint union of simple closed curves (knots) in 3-dimensional space. A *knot* is a link with one component. Links can be represented by projections onto the plane, called *link diagrams*, which may be viewed as plane graphs whose vertices are the under/over crossings. (Refer to (Rolfsen, 1976) for basic information about knots and links).

The main problems of knot theory are: 1) to decide whether a knot is really knotted, 2) decide when two knots are equivalent, and 3) "classify" all possible knots. We make the notion of equivalent links more precise below.

Definition 1. Two links are said to be *equivalent* if there exists an isotopic deformation of the underlying 3-space which carries one link to the other.

In other words, two links are equivalent if there exists an *ambient isotopy*.

Theorem 1. (Reidemeister) *Two diagrams represent equivalent links if and only if one can be transformed from the other by a finite sequence of Reidemeister moves and by planar isotopy.*

By planar isotopy, we mean motions of the diagram in the plane which preserve the graphical nature of the underlying universe. Reidemeister moves are explained below.

Reidemeister's theorem is a basis for a combinatorial approach to knot theory. In particular, it allows the definition of an invariant of links as a mapping from the set of equivalence classes of diagrams to a set of values which is invariant under the Reidemeister moves.

The three types of Reidemeister moves are shown below. The Type III move (called a star-triangle move) is related to the quantum Yang-Baxter equations of physics.

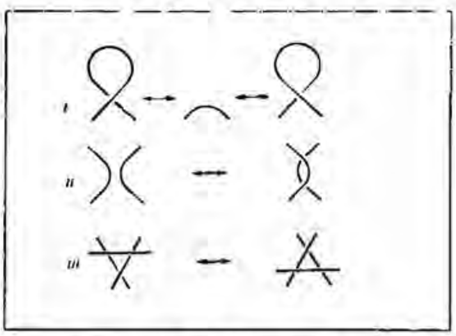

Figure 1. Type I, Type II, and Type III Reidemeister Moves.

An *invariant* is a quantity associated with a link or knot which depends only on the knot in 3-dimensions. For example, the *number of components* is an invariant, although not particularly a strong one. Another simple invariant is the *linking number,* which gives an idea of how two knots wrap around each other, and requires a precise definition of orientation and signs.

An *Ω-valued invariant* for oriented links is a map which associated to each oriented link an element of some ring Ω, for example, the complex field, or a ring of Laurent polynomials, in such a way that equivalent links have the same images in Ω. Classically, one of the most studied examples of link invariants is the *Alexander-Conway polynomial* defined by J. W. Alexander in 1928. Traditionally, these polynomials were understood only in terms of standard algebraic topology, until the advent of Jones' work.

After the discovery of the Jones polynomial, other polynomial invariants were soon discovered. The new invariants provide powerful methods for distinguishing inequivalent knots. See (Kauffman, 1988) for a survey. The most striking feature of these new invariants, aside from their often mysterious properties, is their (previously unheard of) connection with other branches of mathematics, as well as the natural sciences, in particular, to statistical mechanics in physics, and to molecular biologists studying DNA.

SPIN MODELS AND ASSOCIATION SCHEMES

We first give the definition of commutative association schemes. For more information on association schemes, see (Bannai-Ito, 1984).

Definition 2. A *commutative association scheme with d classes,* denoted $\mathcal{X} = (X, \{R_i\}\ 0 \le i \le \mathrm{d})$, consists of a nonempty finite set X together with subsets $\mathrm{R}_i \subseteq \times X$ $(i = 0, 1, \ldots, d)$ satisfying the following axioms:

1. $\mathrm{R}_0 = \{(x, x) | x \in X\}$,
2. $X \times X = \cup_{i=0}^{d} R_i$, $\quad R_i \cap R_j = \emptyset$ if $i \neq j$,
3. ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^tR_i = \{(x, y) | (y, x) \in R_i\}$,
4. For $i, j, k \in \{0, 1, \ldots, d\}$, the integer $p_{ij}^k := \#\{z \in X | (x, z) \in R_i$ and $(z, y) \in R_j\}$ is constant whenever $(x, y) \in R_k$, and
5. $p_{ij}^k = p_{ij}^k$ for all $i, j, k \in \{0, 1, \ldots, d\}$.

In (Jaeger, 1992) the Bose-Mesner algebras generated by the adjacency matrices of some special association schemes were studied and were discovered to be natural places to look for spin models. In particular, Jaeger proves the following result which has since been generalized by others. For instance, see (Bannai-Bannai, 1993).

Theorem 2. (Jaeger) *Let (X, W_+, W_-) be a symmetric spin model. Let $\mathcal{M}$ be the algebra defined by W_+ and the all 1-matrix J with ordinary matrix multiplication. If $\mathcal{M}$ is also closed under entry-wise Hadamard multiplication, then $\mathcal{M}$ is the Bose-Mesner algebra of a self-dual association scheme.*

We now give the definition of a spin model. In this paper, we will use the generalized version of a spin model (formulated by Y. Watatani in (Kawagoe-Munemasa-Watatani, 1994) who dropped the symmetry condition upon the suggestion of Jones).

Definition 3. Let X be a finite set with n elements and w_+ and w_- be complex-valued functions on $X \times X$. Let W_+ and W_- be the matrices of size $n \times n$ defined by $W_+ = (w_+(a,b))_{a,b \in X}$ and $W_- = (w_-(a,b))_{a,b \in X}$. If the following conditions are satisfied, the configuration(X, w_+, w_-) is called a *spin model:*

(1) $^tW_+ \circ W_- = J$, the matrix whose entries are all 1,

(2) $W_+W_- = nI$, where I is the identity matrix of size n,

(3) $^tW_+Y_{ac} = \sqrt{n}w_-(a,c)Y_{ac}$ for any $a, c \in X$.

In the above definition, Y_{ac} is the column vector whose x-entry is given by $(Y_{ac})_x = (w_+(a,x)w_-(x,c))_{x \in X}$. Hadamard or entry-wise multiplication is denoted by o.

This definition generalizes Jones' original definition, where he requires that the matrices W_+ and W_- are both symmetric. We refer to spin models satisfying this additional condition as *symmetric spin models.*

Jones showed how a (symmetric) spin model may be used to obtain a link invariant. Watatani, Munemasa, and Kawagoe show similarly that the generalized version of a spin model provides an invariant of oriented links. We describe the procedure briefly.

Invariants of Oriented Links

For any connected diagram L of an oriented link, we construct a signed graph as follows. Color the regions in black and white so that the unbounded region is white and adjacent regions have different colors. The set of vertices of the graph is the set of black regions, and the set of edges is the set of crossings. An edge is incident with a vertex if and only if the corresponding crossing is on the boundary of a corresponding black region. We assign signs + and –, and an orientation to each edge according to the rules given in Fig. 2 and Fig. 3.

If e is an edge, then we denote the sign of e by $e(e)$, the initial vertex of e by i_e, and the terminal vertex by t_e. Let υ(L) be the number of black regions. If (W_+, W_-) is a generalized spin model on X, then the partition function is defined by

$$Z(L) = \sqrt{|X|}^{\,\upsilon(L)} \sum_{\sigma} \prod_{e} w_{e(c)}(\sigma(i_e), \sigma(t_e)),$$

where the product is taken over all edges e, and the sum is taken over all mappings σ from the set of vertices to X. It is then shown in (Kawagoe-Munemasa-Watatani, 1994) that the partition function above is invariant under Reidemeister moves of types II and III, as long as the diagrams are connected. Moreover, by a suitable normalization factor (which depends on the writer of L), we obtain an invariant of the oriented link L.

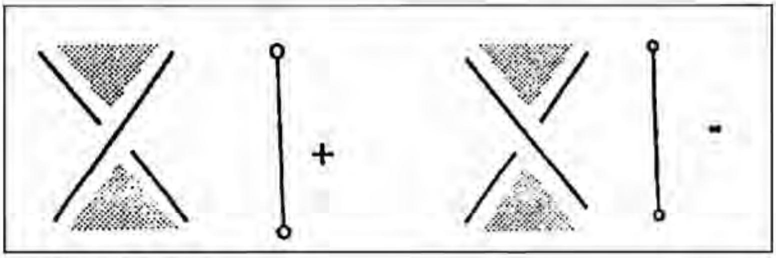

Figure 2. Sign of an Edge

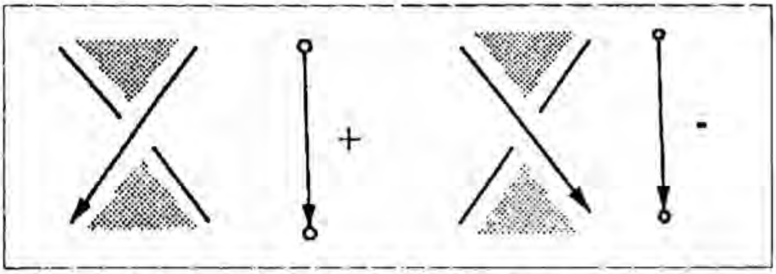

Figure 3. Orietation of an Edge

Classifying Spin Models Via Association Schemes

As stated in the introduction, the current goal is to obtain a complete classification of spin models in terms of association schemes. The simplest case is that of the Potts model for the Jones polynomial, which corresponds to the 1-class association scheme (i.e., the complete graphs). We describe this below.

Example. (*Potts model,* cf (Jones, 1989) and (de la Harpe-Jones, 1993)) Let X be a set of cardinality n. Let the relations be given by $R_0 = \{(x,x)|x \in X\}$ and $R_1 = \{(x,y)|x \neq y \text{ and } x,y \in X\}$. Then $(X, \{R_0, R_1\})$ is a symmetric association scheme of class 1. (Note that (X, R_1) is the complete graph K_n). The Bose-Mesner algebra $\mathcal{A}$ is given by

$$A = (A_0, A_1) = (I,J)$$

Where $A_0 = I$ and $A_1 = J - I$. Let

$$W_+ = t_0 A_0 + t_1 A_1 \text{ (and hence) } W_- = t_0^{-1} A_0 + t_1^{-1} A_1$$

with t_0 and t_1 satisfying

$$t_1^2 + t_1^{-2} + D = 0 \ \ (D^2 = n) \text{ and } t_0 = -(t_1)^{-3}.$$

Then (X, W_+, W_-) is a spin model, called the *Potts model*. We note that the link invariant obtained from this model is a special value of the Jones polynomial for oriented links.

Jones' work suggested that the sources of symmetric spin models seen to be the 2-class association schemes some special regularity conditions (e.g. self-dual and locally strongly regular). Contained in this class are the symmetric conference graphs, which include the family of Paley graphs. In the paper (Balmaceda, 1993), the author proves the following theorem:

Theorem 3. Let *X be a set of size n, and* (X, W_+, W_-) *be a symmetric spin model associated with a symmetric conference graph G with* $n \geq 5$ *which is not a Potts model. Then G is either the pentagon or the lattice graph* $L_2(3)$ *with nine vertices.*

By a symmetry conference graph, we mean a strongly regular graph with parameters $(n, k, \lambda, \mu) = (n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-5}{4})$. The natural extension of the above result would be to consider the non-symmetric conference digraphs (i.e. non-symmetric association schemes of class 2), and to determine if one obtains a generalized spin model from such schemes.

In the next section, we give a direct proof of the following result: there are no such spin models, except when $n = 3$. For the case $n = 3$, Munemasa and Watatani have previously found some examples (Kawagoe-Munemasa-Watatani, 1994).

RESULTS AND PROOFS

Let $\mathcal{X} = (X, \{R_i\} 0 \leq i \leq 2)$ be a non-symmetric 2-class association scheme. Then it is known that $|X| = n = 4m + 3$, for some integer m and that $\mathcal{X}$ is self-dual. Furthermore, if A_0, A_1, A_2 are the adjacency matrices of the scheme, $A_0 = I$ and $A_1^t = A_2$. We display the intersection matric B and character table P of such schemes below. P is the matrix with entries $(P)_{i,j} = (p_j(i)), 0 \leq i,j \leq 2$.

$$B = \begin{pmatrix} p_{10}^0 & p_{10}^1 & p_{10}^2 \\ p_{11}^0 & p_{11}^1 & p_{11}^2 \\ p_{12}^0 & p_{12}^1 & p_{12}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & m+1 \\ 2m+1 & m & m \end{pmatrix}$$

$$P = \begin{pmatrix} p_0(10) & p_1(0) & p_2(0) \\ p_0(10) & p_1(0) & p_2(0) \\ p_0(10) & p_1(0) & p_2(0) \end{pmatrix} = \begin{pmatrix} 1 & \frac{n-1}{2} & \frac{n-1}{2} \\ 1 & \frac{-1+\sqrt{ni}}{2} & \frac{-1-\sqrt{ni}}{2} \\ 1 & \frac{-1-\sqrt{ni}}{2} & \frac{-1+\sqrt{ni}}{2} \end{pmatrix}$$

Let $W_+ = t_0A_0 + t_1A_1 + t_2A_2$, for nonzero complex constants t_0, t_1, and t_2. Following the observation in (Kawagoe-Munemasa-Watatani, 1994), we may assume that $t_0 = t_1 \neq t_2$; otherwise, we will obtain the (trivial) case of the Potts model. Hence we have, by the first condition in the definition of spin models:

$$W_- = t_0^{-1}A_0^t + t_1^{-1}A_1^t + t_2^{-1}A_2^t = t_1^{-1}A_0 + t_1^{-1}A_2 + t_2^{-1}A_1.$$

A useful ingredient in the proof is the result of Jaeger ([6], Prop. 5) which we state below:

Lemma 4. *Let W_+ and W_- be as above. Then (X, W_+, W_-) is a spin model if and only if the following hold:*

(1) $PT = \sqrt{n}T^{-1}$, where $T = (t_0, t_1, t_2)^t$ and $T^{-1} = (t_0^{-1}, t_1^{-1}, t_2^{-1})^t$

(2) $E_iY_{ab} = 0$ for every $i, j \in \{1, 2\}$ with $t_{i'} \neq t_j$, and for every pair $(a, b) \in R_j$.

In the lemma above, the matrices E_i are the primitive idempotents of the scheme chosen in a suitable ordering.

We now state and prove the following result.

Theorem 5. *There are no spin models arising from the nonsymmetric confdrence digraphs, except the ones corresponding to the Potts models and the one on three vertices.*

Proof. Since the schemes in consideration have size $n = 4m + 3$, the smallest ones correspond to $m = 0$, i.e., on $n = 3$ vertices. And Munemasa and Watatani have found an example corresponding to this case. Moreover, we saw earlier that the Potts models arise (in a trivial way) from the series of graphs being considered

We now show that for $m > 0$, no other spin models can be found. It is enough to show that condition (2) of the lemma is not satisfied. The proof utilizes the algebraic and combinatorial properties of spin models and the corresponding properties of the association schemes involved.

We now proceed with our computations. We assume that $(a, b) \in R_1$. Then, since $t_{1'} \neq t_1$, it is enough to show that $E_1Y_{ab} \neq 0$, the zero vector.

Since $E_1 = \frac{1}{n}\sum_{j=0}^{2}\overline{p_1(j)}\,A_j$, we see that the vector $E_1Y_{ab} = 0$ if and only if $\sum_{j=0}^{2}\overline{p_1(j)}\,A_jY_{ab} = 0$. Let $(A_j)x,y$ and $(\upsilon)_y$ be the (x, y)-entry of A_j and the y-entry of υ, respectively, where υ is a column vector of size n, and $x, y \in X$.

Then

$$(A_jY_{ab})_x = \sum_{y\in X}(A_j)_{x,y}(A_{ab})_y = \sum_{y\in X}(A_j)_{x,y}\,w + (a, y)w - (y, b).$$

Since $w_+(a,y) = t_k$ when $(a, y) \in R_k$ and $w_-(y, b) = t_k^{-1}$ when $(y, b) \in R_k$, we have

$$(A_j)_{x,y}\,(Y_{ab})_y = \sum_{k,l=0}^{2} p_{jkl}\,(x, a,b)\; t_kt_y^{-1},$$

where $p_{jkl}(x, a, b) = \#\{y|(x,y) \in R_j\,(a, y) \in R_k, (y, b) \in R_l\} := p_{jkl}$.

Suppose that $x \in X$ satisfies the following: $(x, a) \in R_2$, $(x, b) \in R_2$, with $(a, b) \in R_1$. We now show that the equation $E_1Y_{ab} = 0$ is not satisfied for the above choice of x. For this we need to compute all the values of the intersection numbers p_{jkl}, $0 \leq j,k,l \leq 2$.

Case j = 0: It is straightforward to compute the following values: $p_{100} = p_{101} = p_{002} = p_{101} = p_{011} = p_{020} = p_{021} = p_{022} = 0$. The only nonzero value is $p_{012} = 1$.

Case j = 1: Again we can compute directly the following values: $p_{100} = p_{101} = p_{102} = p_{110} = p_{120} = 0$. We now compute the rest of the values by analyzing the parameters of nonsymmetric conference digraphs.

Suppose $p_{111} = \dot{r}$. Since $p_{11}^2 = \#\{y|(x,y) \in R_1, (y,b) \in R_1\}$ for $(x, b) \in R_2$, we get:

$$m + 1 = \sum_{i=0}^{2}\; p_{1i]}p_{101} + p_{111} + p_{111}.$$

Thus,

$$m + 1 = 0 + r + p_{121} \quad \text{ i.e., } p_{121} = m - r + 1.$$

On the other hand, we also have

$$p_{11}^2 = \#\{y|(x, y) \in R_1, (y, a) \in R_1\} \quad \text{for } (x, a) \in R_2,$$
$$= \#\{y|(x, y) \in R_1, (y, a) \in R_2\} \quad \text{for } (x, a) \in R_2.$$

Thus

$$p_{11}^2 = \sum_{i=0}^{2}\; p_{12i} = p_{120} + p_{121} + p_{122}.$$

Or,

$$m + 1 = 0 + m - r + 1 + p_{122}, \quad \text{i.e., } p_{122} = r.$$

Finally,

$$p^2_{12} = \#\{y|(x, y) \in R_1, (y, b) \in R_2\} \text{ where } (x, b) \in R_2.$$

So,

$$m = \sum_{i=0}^{2} p_{1i2} = p_{102} + p_{112} + p_{122}.$$

Or,

$$m = 0 + p_{112} + p_{122} \text{ Hence, } p_{112} = m - p_{122} = m - r.$$

Summing up, we have: $p_{111} = r$, $p_{112} = m - r$, $p_{121} = m - r + 1$, $p_{122} = r$, and $p_{100} = p_{102} = p_{101} = p_{120} = 0$.

Case j = 2. As before, it is straightforward to compute the following values: $p_{200} = p_{202} = p_{220} = 0$. Similarly, $p_{201} = p_{210} = 1$. We still need to compute: p_{211}, p_{212}, p_{221}, and p_{222}.

Suppose $_{211} = s$. Then

$$p^2_{21} = \#\{y|(x, y) \in R_2, (y, b) \in R_1\} \text{ where } (x, b) \in R_2.$$

So,

$$m = \sum_{i=0}^{2} p_{2i1} = p_{201} + p_{211} + p_{221} = 1 + s + p_{221}.$$

Hence,

$$p_{221} = m - s - 1.$$

Now

$$p^2_{21} = \#\{y|(x, y) \in R_2, (y, a) \in R_1\} \quad \text{for } (x, a) \in R_2,$$
$$= \#\{y|(x, y) \in R_2, (a, y) \in R_2\} \quad \text{for } (x, a) \in R_2.$$

So,

$$m = p^2_{21} = \sum_{i=0}^{2} p_{22i} = p_{220} + p_{221} + p_{222} = 0 + (m - s - 1) + p_{222}.$$

Thus,

$$p_{222} = s + 1.$$

Similarly,

$$p^2_{22} = \#\{y|(x, y) \in R_2, (y, b) \in R_2\} \quad \text{when } (x, a) \in R_2.$$

So,

$$m = \sum_{i=0}^{2} p_{2i2} = p_{202} + p_{212} + p_{222} = 0 + p_{212} + (s + 1).$$

Hence,

$$p_{212} = m - s - 1.$$

Finally,

$$p_{11}^1 = \#\{y|(a, y) \in R_1, (y, b) \in R_1\} \text{ for } (a, b) \in R_1.$$

So,

$$m = \sum_{i=0}^{2} p_{i11} = p_{011} + p_{111} + p_{211} = 0 + r + s.$$

Hence,

$$m = s - r.$$

And thus:

$$p_{211} = m - r, \; p_{212} = r - 1, \; p_{221} = r - 1, \; p_{222} = m - r.$$

To complete the proof, we now show that the second condition of the lemma does not hold. Recall that

$$(E_1 Y_{ab})_x = 0 \Leftrightarrow \sum_{j=0}^{2} \overline{p_i(j)}. \sum_{k,l=0}^{2} p_{jkl} t_k t_y^{-1} = 0$$

Then, substituting the values obtained for pijk,

$$\sum_{j=0}^{2} \overline{p_i(j)}. \sum_{k,l=0}^{2} p_{jkl} t_k t_y^{-1} = \overline{p_i(0)} t_1 t_{2'}^{-1}$$

$$+ \overline{p_i(1)} [r t_1 t_{1'}^{-1} + (m - r) t_1 t_{2'}^{-1} + (m - r + 1) t_2 t_{1'}^{-1} + r t_2 t_{2'}^{-1}]$$

$$+ \overline{p_i(2)} [t_0 t_{1'}^{-1} + t_1 t_0^{-1} + (m - r) t_1 t_{1'}^{-1} + (r - 1) t_1 t_{2'}^{-1} + (r - 1) t_2 t_{1'}^{-1}$$

$$+ (m - r + 1) t_2 t_{1'}^{-1}].$$

Since $t_0 = t_1$, $t_{1'} = t_2$, and $t_2 = t_{1'}$, and using the values of the $p_i(j)$ obtained from the character table P of the scheme, the right-hand side of the above equation simplifies to the following:

$$2m + 1) + \frac{-1-\sqrt{ni}}{2} [r \frac{t_1}{t_2} + (m - r) + (m - r + 1) r \frac{t_2}{t_1}]$$

$$+ \frac{-1 + \sqrt{ni}}{2} [\frac{t_1}{t_2} + 1 + (m - r) \frac{t_1}{t_2} + (r - 1) + (r - 1) + (m - r + 1) \frac{t_2}{t_1}]$$

Collecting terms and simplifying we obtain

$$[\frac{t_1}{t_2} + \frac{t_2}{t_1} - 2] [\frac{-(m+1)}{2} + \frac{(m - 2r + 1)\sqrt{n}}{2}]$$

Since $t_1 + t_2 \neq 0$, the factor $\frac{t_1}{t_2} + \frac{t_1}{t_2} - 2$will never equal zero. Moreover, the second factor is clearly nonzero. Hence we have shown that $(E_1 Y_{ab})_x \neq 0$. From the lemma, we conclude that no spin model arises from this class of association schemes.

Finally, we note that for the second possible ordering of the primitive idempotents E_i, which results in reversing the second and third rows of the character table P, a similar argument yields the same conclusion as before. This completes the proof. □

CONCLUDING REMARKS

The relation between knot theory and statistical mechanics and the role played by association schemes are not yet very well understood, but the evidence for the connection is substantial. Indeed a real understanding will require the efforts of both mathematicians and physicists, working in diverse areas such as topology, quantum field theory, combinatories and the geometry of 3-manifolds.

Through the years, many mathematical theories and objects have found relevance and applications in otherwise unrelated areas. The new knot theory, for instance, has been used by molecular biologists in enzyme recognition and the study of DNA, and by chemists studying polymer theory. It is hoped that this article serves as introduction and encouragement for interdisciplinary research.

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